

STRONG EDGE-COLORINGS FOR k -DEGENERATE GRAPHS

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ABSTRACT. We prove that the strong chromatic index for each k -degenerate graph with maximum degree Δ is at most $(3k - 1)\Delta - k(k + 1) + 1$.

A *strong edge-coloring* of a graph G is an edge-coloring so that no edge can be adjacent to two edges with the same color. So in a strong edge-coloring, every color class gives an induced matching. The strong chromatic index $\chi'_s(G)$ is the minimum number of colors needed to color $E(G)$ strongly. This notion was introduced by Fouquet and Jolivet (1983, [5]). Erdős and Nešetřil during a seminar in Prague in 1985 proposed some open problems, one of which is the following

Conjecture 1 (Erdős and Nešetřil, 1985). *If G is a simple graph with maximum degree Δ , then $\chi'_s(G) \leq 5\Delta^2/4$ if Δ is even, and $\chi'_s(G) \leq (5\Delta^2 - 2\Delta + 1)/4$ if Δ is odd.*

This conjecture is true for $\Delta \leq 3$ ([1, 6]). Cranston [4] showed that $\chi'_s(G) \leq 22$ for $\Delta = 4$. Chung, Gyárfás, Trotter, and Tuza (1990, [3]) showed that the upper bounds are exactly the numbers of edges in $2K_2$ -free graphs. Molloy and Reed [8] proved that graphs with sufficient large maximum degree Δ has strong chromatic index at most $1.998\Delta^2$. For more results see [9] (Chapter 6, problem 17).

A graph is k -degenerate if every subgraph has minimum degree at most k . Chang and Narayanan (2012, [2]) recently proved that a 2-degenerate graph with maximum degree Δ has strong chromatic index at most $10\Delta - 10$. There is a gap in their proof, which was fixed by Luo and the author in [7], and they actually improved the upper bound to $8\Delta - 4$.

In this short paper, we study the strong chromatic index for k -degenerate graphs. Unlike the priming processes in [2, 7], we find a special ordering of the edges and by a greedy coloring, the following better and more general result is obtained

Theorem 1. *The strong chromatic index for each k -degenerate graph with maximum degree Δ is at most $(3k - 1)\Delta - k(k + 1) + 1$.*

Thus, 2-degenerate graphs have strong chromatic index at most $5\Delta - 5$.

Proof. First of all, we will need the following simple fact on k -degenerate graphs (see [2]).

Let G be a k -degenerate graph. Then there exists $u \in V(G)$ with $d(u) > k$ so that u is adjacent to at most k vertices of degree more than k .

We call a vertex u with degree more than k a *special vertex* if u is adjacent to at most k vertices of degree more than k . An edge is a *special edge* if it is incident to a special vertex and a vertex with degree at most k . The above fact implies that every k -degenerate graph has a special edge.

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We order the edges of G as follows. First we find in G a special edge, put it at the beginning of the list, and then remove it from G . Repeat the above step in the remaining graph. When the process ends, we have an ordered list of the edges in G , say e_1, e_2, \dots, e_m , where $m = |E(G)|$. So e_m is the special edge we first chose and placed in the list.

Let G_i be the graph induced by the first i edges in the list, $i = 1, 2, \dots, m$. Then e_i is a special edge in G_i , by the above construction. So in G_i , e_i has at most

$$k\Delta + (k-1)\Delta + k(\Delta - k - 1) = (3k-1)\Delta - k(k+1)$$

edges within distance 1 which are in G_i .

Now color the edges in the list one by one greedily. For each i , when it is the turn to color e_i , only the edges in G_i have been colored. Since there are at least $(3k-1)\Delta - k(k+1) + 1$ colors, we are able to color the edges so that edges within distance 1 get different colors. \square

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